

NONSTATIONARY PROCESSES IN A ONE-DIMENSIONAL
CHAIN OF COUPLED ANHARMONIC OSCILLATORS

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The development of vibrations in a one-dimensional chain of coupled oscillators is studied. Under the assumption that the interaction between oscillators is weakly nonlinear the methods of nonlinear mechanics are used to find the time dependence of the displacements of elements of the chain after an initial displacement is given one of the links. The main features of the development of vibrations are shown. A comparison is made with the results of the linear theory.

Coupled oscillators simulate various mechanical, electrical, and physical systems. In theoretical papers treating nonstationary vibrations in systems of coupled oscillators [1-4] the interaction was assumed harmonic. However, the linear approximation of the interaction is only the simplest approximation. In a number of cases the inadequacy of the linear theory is evident. Numerical calculations [5-7] and papers taking account of the nonlinearity of the interaction for particular elements of the structure [6-8] only partially fill the gap. We discuss below the problem of finding the laws of motion of an infinite one-dimensional chain, taking account only of nearest-neighbor interactions with a weak nonlinearity.

If at time zero a certain element of the chain, distinguished by the subscript zero, receives a displacement a_0 , the solution of the linear problem has the form

$$x_n(t) = a_0 I_{2n}(\omega t), \quad (1)$$

where $x_n(t)$ is the displacement of the n -th element from its equilibrium position, I_i is the i -th order Bessel function of real argument, ω is the cutoff frequency of vibrations of the chain, t is the time, and the links are numbered in order from the "zero" element.

Following the basic premises of the asymptotic methods of studying nonlinear vibrations [9-10] we seek the solution of the problem in the form

$$x_n(t) = a_n(t) I_{2n}(\omega t + \theta_n), \quad (2)$$

where a_n and θ_n are functions of the time to be determined. It is expeditious to introduce them in the following way. We require that the expressions for the time derivatives of the displacements x_n have a form analogous to the corresponding linear expression, i.e.,

$$dx_n/dt = \frac{\omega}{2} (I_{2n-1} - I_{2n+1})(\psi_n) \cdot a_n, \quad \psi_n = \omega t + \theta_n. \quad (3)$$

Differentiating (2) with respect to t and taking account of the postulated relation (3) we obtain

$$\frac{da_n}{dt} = -2 \left(\frac{I_{2n}}{I_{2n-1} - I_{2n+1}} \right) (\psi_n) a_n^{-1} \frac{da_n}{dt}. \quad (4)$$

The equations of motion of the elements of the structure have the form

$$d^2x_n/dt^2 = \omega^2/4 (x_{n-1} - 2x_n + x_{n+1}) + \varepsilon [f(x_{n-1} - x_n) - f(x_n - x_{n+1})], \quad (5)$$

where f is a function defining the form of the nonlinearity and ε is a small parameter.

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Equating the right-hand sides of Eqs. (5) to the result of differentiating Eqs. (3) with respect to the time we obtain a set of equations relating a_n and θ_n . By using (4) we obtain

$$\begin{aligned} da_n/dt = & 1/2\omega (I_{2n-1} - I_{2n+1})(\Psi_n) \{ \varepsilon f [a_{n-1} I_{2n-2}(\Psi_{n-1}) - a_n I_{2n}(\Psi_n)] - \\ & - \varepsilon f [a_n I_{2n}(\Psi_n) - a_{n+1} I_{2n-2}(\Psi_{n+1})] + \omega^2/4 [a_{n-1} I_{2n-2}(\Psi_{n-1}) + \\ & + a_{n+1} I_{2n+2}(\Psi_{n+1}) - a_n (I_{2n-2} + I_{2n+2})(\Psi_n)] \} \{ 1/4 [(I_{2n-1} - I_{2n+1})^2 - I_{2n} (I_{2n-2} - 2I_{2n} + I_{2n+2})] (\Psi_n) \}^{-1}. \end{aligned} \quad (6)$$

The first-order equations (4) and (6) for the new functions a_n and θ_n are equivalent to the original system of equations (5).

Since da_n/dt is different from zero as a direct consequence of the presence of the nonlinear contribution, it may seem strange that the right-hand sides of Eqs. (6) have terms which do not contain ε . It should be taken into account however, that the difference between a_i and a_j , just as between Ψ_i and Ψ_j ($i \neq j$), results from the presence of the nonlinearity. In the linear case these quantities are the same for all subscripts. If $\varepsilon \neq 0$ there is generally no basis for assuming that the relations $a_n(t)$ are the same for all n . Thus as $\varepsilon \rightarrow 0$ $a_{n+1}, a_{n-1} \rightarrow a_n$ and $\Psi_{n+1}, \Psi_{n-1} \rightarrow \Psi_n$, which ensures the transition to the original linear case.

Equations (4) and (6) are no simpler than (5). The exact system (4), (6) can, however, be replaced by a simpler approximation. First of all, in the terms which are proportional to ε the difference between a_n and $a_{n \pm 1}$ and between Ψ_n and $\Psi_{n \pm 1}$ can be neglected. The error in this simplification is of a higher order of smallness than ε . Secondly, Eqs. (4) and (6) can be averaged over a range of Ψ characteristic for the Bessel functions. The averaging operation will affect only the known functions. We change from the variables a_n and θ_n to the corresponding quantities averaged over the characteristic vibration cycle and denote the latter by the same symbols as for the corresponding unaveraged quantities. The procedure described is analogous to the standard technique of separating the "fast" and "slow" times, i.e., the asymptotic methods of studying nonlinear vibrations [9-10]. The averaging operation is particularly simple for large values of the arguments of the Bessel functions (for large times) since in this asymptotic region the lengths of the vibration cycles described by the Bessel functions are unchanged. The infinite set of nonlinear first-order equations obtained by the operations described cannot be solved in general form. However, a solution can be found in certain interesting cases.

The situation is simplest when the function f is odd, i.e., when the curve for the dependence of the potential energy on distance is symmetric. Using the asymptotic expressions for the Bessel functions [11]

$$I_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left\{ \cos \left(z - \frac{\pi \nu}{2} - \frac{\pi}{4} \right) - \frac{4\nu^2 - 1}{8z} \sin \left(z - \frac{\pi \nu}{2} - \frac{\pi}{4} \right) \right\}, \quad (7)$$

which are accurate to terms $\sim z^{-1}$, we assume that $f(y)$ corresponds to some odd power of y ($f(y) = y^{2m+1}$). We also assume that $a_{n+1} = a_{n-1} = a_n$ and $\theta_{n+1} = \theta_{n-1} = \theta_n$. The validity of these relations is not obvious, but in the range of applicability of Eq. (7) they are confirmed by calculations.

To the accuracy of Eq. (7) the denominator of Eq. (6) is $2(\pi\psi)^{-1}$. Averaging over a 2π interval gives

$$\frac{da_n}{dt} = -2^{3m+1} \frac{\varepsilon}{\omega} \frac{1}{\pi^m} \frac{(2m+1)!!}{(2m+2)!!} \frac{a_n^{2m+1}}{(\omega t)^{m+1}}. \quad (8)$$

The solution of the nonlinear equation (8) for $m \neq 0$ has the form

$$\begin{aligned} a_n(t) &= a_{n0} \left\{ \frac{z_0^m}{z_0^m + 2Aa_{n0}^{2m} (z^m - z_0^m)} \right\}^{\frac{1}{2m}}; \\ z &= \omega t, \quad z_0 = \omega t_0, \quad a_{n0} = a_n(t_0); \\ A &= 2^{2m} \frac{\varepsilon}{\omega} \frac{1}{\pi^m} \frac{(2m+1)!!}{(m+1)!}. \end{aligned} \quad (9)$$

The reference point t_0 must be chosen so that at that time the Bessel functions are adequately approximated by the asymptotic forms (7). For small enough ε and n the value of a_{n0} hardly differs from $a_n(0) = a_0(0)$.

The above assumptions are confirmed by the fact that the amplitude factors a_n do not depend on n . The quantities $a_n(t)$ increase or decrease monotonically depending on the sign of ε . For $\varepsilon > 0$ a_n decreases, and for $\varepsilon < 0$ a_n increases. In both cases a_n approaches the limit

$$a_{n0} \left\{ \frac{z_0^m}{z_0^m + 2Aa_{n0}^{2m}} \right\}^{\frac{1}{2m}}. \quad (10)$$

The decrease of the amplitude of the vibrations of the n-th element as compared with the linear case (1) for those same times when $\varepsilon > 0$ is explained by the increase in the effective velocity of sound with increasing effective stiffness of the structure, i.e., the increase in the rate of removal of energy from the region of the initial perturbation.

The development of vibrations in the chain for $\varepsilon > 0$ can be represented in the following way. The front of the elastic wave moves with a higher velocity than in the corresponding linear case. After the passage of the front of the elastic wave the amplitude of the vibrations of these elements is greatly decreased, and according to Eq. (9) there is a more rapid damping of the vibrations in the asymptotic regime. When the amplitude of the vibrations has decreased to a certain value the development of the process goes over to the "linear regime" corresponding to the maximum amplitude factor (10). The transition stage to the regime of linear vibrations is shorter the larger m, that is, the steeper the edge of the potential well.

The equation for averaging the phase correction θ_n has the form

$$\frac{d\theta_n}{dt} = 2A a_n^{2m} (\omega t)^{-m}. \quad (11)$$

Hence it follows that

$$\theta_n = \theta_{n0} + 2A \frac{a_{n0}^{2m}}{(m-1)\omega} \frac{z^{m-1} - z_0^{m-1}}{(z_0)^{m-1}}, \quad m \neq 0, 1; \quad (12)$$

$$\theta_n = \theta_{n0} + 12 \frac{\varepsilon}{\pi \omega^2} a_0^2 \ln \frac{t}{t_0}, \quad m = 1; \quad (13)$$

$$\theta_n = \theta_{n0} + 2 \frac{\varepsilon}{\omega} (t - t_0), \quad m = 0; \quad (14)$$

$$\theta_{n0} = \theta_n(t_0).$$

The notation in Eqs. (11)-(14) is the same as in (9). In integrating Eq. (11) it is assumed that $a_n = a_{n0}$, since the difference between a_n and a_{n0} is of the order ε .

Equation (12) determines the speeding up of the phase growth in comparison with a linear system for $\varepsilon > 0$ and the slowing down for $\varepsilon < 0$. The limiting value of θ_n depends critically on the amplitude of the vibrations, characterized by a_{n0} .

When $m=0$ and $m=1$ the phase correction θ_n does not have a limit. When $m=1$ $d\theta_n/dt$ decreases without bound as t increases, and for $m=0$ $d\theta_n/dt = \text{const} = 2\varepsilon\omega^{-1}$. The result for $m=0$ follows directly from the form of the equations of motion (5) and is of interest only as an illustration of the possibility of the method.

It must be supposed that Eqs. (11)-(14) for θ_n are more accurate than Eqs. (8) and (9) for a_n since the form of Eq. (11) is determined by the contribution of the dominant terms of Eq. (7) and not by the correction terms ($\sim z^{-3/2}$) as in the equations for a_n .

Within the framework of the method used the most favorable case is $m=0$. Here, in contrast with other cases with $m > 0$, the value of θ_n is finite for all t . Thus in this case the postulated relations (3) and (4) are not completely fortunate. If Eq. (3) corresponds to the actual properties of the structure for $m > 0$ and sufficiently large t , for $m=0$ such a relation is determined only by the method of calculation; i.e., in introducing condition (3) we perform a certain renormalization of the amplitude and phase corrections. For $m=0$ this is somewhat inconvenient. Nevertheless the result (14) corresponds completely to the properties of the structure. This fact increases the confidence in the reliability of the results found in the present paper for nonlinear systems.

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